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Construction scheme for discrete Miura transformations

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Abstract. A direct and elementary scheme for the construction of Miura-type transformations and discrete differential equations related to them (scalar and vector) is presented. The scheme is illustrated using as examples the Volterra and Toda models. A discrete-differential analogue of the Calogero–Degasperis equation is discussed in detail. This example is used to show how to construct conservation laws, higher symmetries, and solutions for an equation obtained with the help of the scheme.

1. Introduction

A construction scheme for Miura-type transformations of partial differential equations was presented in [1] (analogous problems were discussed in [2]). In [1] the situation is discussed in which two equations (F) and (G) are reduced to a third one (T) by Miura-type transformations (these transformations have a special form):

$$\begin{array}{ccc} (F) & \longleftarrow & (H) \\ \downarrow & & \downarrow \\ (T) & \longleftarrow & (G) \end{array} . \qquad (1)$$

It is shown that there exists an equation (H) which can be reduced to (F) and (G) by transformations of the same type. It is explained how to construct (H) and corresponding transformations. If the given equations (F), (G), (T) possess conservation laws and symmetries, the new equation (H) will have them too. There are many instances in which the scheme can be applied.

The simplest example is the Korteweg–de Vries equation $\tilde{u}_t = \tilde{u}_{xxx} + 6\tilde{u}\tilde{u}_x$. As the equation (F), take the modified Korteweg–de Vries equation (MKdV equation) $u_t = u_{xxx} - 6u^2 u_x$ related to the Korteweg–de Vries equation by the well known Miura transformation $\tilde{u} = u_x - u^2$ [3]. The equation (G) and corresponding transformation are the MKdV equation and Miura transformation again, but with v in place of u. Let us impose the constraint $u_x - u^2 = v_x - v^2$ and rewrite it as follows: $(u-v)_x = u^2 - v^2$. Now we can introduce a new dynamical variable $w = u - v$, hence $w_x = u^2 - v^2$. The variables u and v can be easily expressed in terms of w, w_x:

$$2u = w^{-1} w_x + w \qquad 2v = w^{-1} w_x - w. \qquad (2)$$

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Differentiating $w=u-v$ with respect to t by virtue of the MKdV equation and using (2), we obtain the equation (H) of (1):

$$r_t=r_{xxx}-\tfrac{1}{2}r_x^3-\tfrac{3}{2}e^{2r}r_x \tag{3}$$

where $r=\ln(w)$. It is not hard to verify that (3) is reduced to the MKdV equation by the Miura-type transformations (2). In other words, the formulae (2) take any solution of (3) into a solution of the MKdV equation. Equation (3) is a particular case of the well known Calogero-Degasperis equation [4]. Miura-type transformation of the Calogero–Degasperis equation into the MKdV equation was found in [5], where integrable Korteweg–de Vries-type equations were classified up to transformations of this kind.

An analogous scheme for discrete differential equations will be discussed in the present paper. As examples we shall consider the Toda model, the Volterra equation, and a discrete differential analogue of equation (3).

2. General schemes

We consider systems of discrete differential equations (chains) of the form

$$(u_n)_t=F(u_{n+k}, u_{n+k-1}, u_{n+k-2}, \ldots, u_{n+m}). \tag{4}$$

Here u_i are vector dynamical variables, F is a vector function, k and m are fixed integers, and n is an integer parameter. The transformations under consideration are of the form $\tilde{u}_n=a(u_{n+1}, u_n)$. Let D be the shift operator which acts on vector functions depending on a finite number of dynamical variables. The shift changes only the subscripts of the variables u_i; for example

$$D(h(u_2, u, u_{-1}))=h(u_3, u_1, u).$$

We see that the chains and transformations under consideration are invariant under the shift. This means, in particular, that any transformation $\hat{u}_n=a(u_{n+i+1}, u_{n+i})$ (i is a fixed integer) can be expressed in the form we consider (one should denote $\hat{u}_n$ by $\tilde{u}_{n+i}$). For the chains (4) the notation $u_{nt}=F[u_n]$ will be used. As a rule, for brevity we shall not write the parameter n. For example, the well known (scalar) Volterra equation $u_{nt}=u_n(u_{n+1}-u_{n-1})$ will be of the form

$$u_t=u(u_1-u_{-1}). \tag{5}$$

One can write down formulae below at $n=0$.

Let us enumerate the conditions sufficient for the scheme to be applied.

Condition 1. There are two chains

$$u_t=F[u] \qquad v_t=G[v] \tag{6}$$

which are reduced to a third one $\tilde{u}_t=T[\tilde{u}]$ (see scheme (1)) by transformations of the form

$$\tilde{u}=a(u_1, u) \qquad \tilde{u}=b(v_1, v). \tag{7}$$

Condition 2. The constraint

$$a(u_1, u) = b(v_1, v) \tag{8}$$

can be expressed in the form

$$D(p(u, v)) = q(u, v) \tag{9}$$

((8) and (9) are equivalent). Coefficients of the vectors p, q are functionally independent.

Condition 1 means, in particular, that the equality

$$a_u F + a_{u_1} D(F) = T[a]$$

holds identically (here a_u, a_{u_1} are Jacobi matrices, for example $a_u = (\partial a^i/\partial u^j)$, where a^i, u^j are coefficients of the vectors a, u). The same identity takes place for b, G. Therefore, differentiating (8) using equations (6), we obtain $T[a] = T[b]$. This relationship is a consequence of (8), i.e. constraint (8) and equations (6) are consistent.

To obtain a new chain, let us consider the system (6), (8) (or (6), (9) which is the same). We shall need the notion of independent dynamical variables. It should be remarked that in the case of (4), for example, the dynamical variables u_i can be regarded as independent. In the case of the system (6), (9), the functions

$$w_n = p(u_n, v_n) \tag{10}$$

can be considered to be independent. The new chain is constructed in terms of the variables w_i. The change of variables (10) is invertible, for (9) implies $w = p(u, v)$ and $w_1 = q(u, v)$, and by condition 2 the variables u, v are expressed via w, w_1:

$$u = r(w_1, w) \qquad v = s(w_1, w). \tag{11}$$

Differentiating $w = p(u, v)$ with respect to t using equations (6) and (11), we easily obtain a chain in terms of w_i:

$$w_t = H[w] = p_u(r, s)F[r] + p_v(r, s)G[s]. \tag{12}$$

If conditions 1 and 2 hold, we construct a new chain by the formula (12). New transformations are given by (11).

It is important that (12) is reduced to (6) by (11). Let us explain why this is true. It follows from (9) that $D(p_u F + p_v G) = q_u F + q_v G$. Comparing this equality with (12), we find

$$\begin{pmatrix} H \\ D(H) \end{pmatrix} = \begin{pmatrix} p_u & p_v \\ q_u & q_v \end{pmatrix} \begin{pmatrix} F[u] \\ G[v] \end{pmatrix}. \tag{13}$$

Note that one can obtain a chain in terms of w_i, using the relationship $w_1 = q(u, v)$. However, as (13) shows, this chain will coincide with (12). The equality (13) is equivalent to

$$\begin{pmatrix} r_w & r_{w_1} \\ \partial_w & \partial_{w_1} \end{pmatrix} \begin{pmatrix} H \\ D(H) \end{pmatrix} = \begin{pmatrix} F[r] \\ G[s] \end{pmatrix}.$$

Taking into account the independence of the dynamical variables w_i, we see that the last equality holds identically. Thus, formulae (11) yield solutions of eqs. (6) for any solution of eq. (12). We are led to the following result.

Theorem. If conditions 1, 2 are valid, then equation (12) is reduced to equations (6) by the corresponding transformations (11).

3. Discrete-differential analogue of the Calogero–Degasperis equation

First the scalar case will be considered, and the scheme will be illustrated using as an example the Volterra equation (5). We shall obtain a discrete differential analogue of the Calogero–Degasperis equation (3) and show how to construct local conservation laws, higher symmetries, and exact solutions for this equation.

3.1. Volterra equation example

The scalar chain (5) is called, owing to its properties, not only the Volterra equation but also the difference KdV equation. It is known that there are discrete differential analogues of both the MKdV equation and the Miura transformation:

$$u_t = (u^2 - \alpha^2)(u_1 - u_{-1}) \tag{14}$$

$$\tilde{u} = (u + \alpha)(u_1 - \alpha) \tag{15}$$

[6–8]. Here (15) reduces (14) to (5) for any constant α. As the chain (G) and corresponding transformation (see scheme (1) and condition 1), use (14) and (15) again, but with v and β instead of u and α. The constraint (8) takes the form

$$(u + \alpha)(u_1 - \alpha) = (v + \beta)(v_1 - \beta) \tag{16}$$

and can be expressed as follows: $(v_1 - \beta)/(u_1 - \alpha) = (u + \alpha)/(v + \beta)$. Condition 2 holds if $\alpha \neq 0$ or $\beta \neq 0$. In accordance with the scheme, the invertible change of variables is defined by $w = (v - \beta)/(u - \alpha)$. It is convenient to carry out the additional point transformation $\tilde{w}_n = (w_n + 1)/(w_n - 1)$:

$$\tilde{w} = (v + u - \eta)/(v - u + \mu) \tag{17}$$

where $\mu = \alpha - \beta$, $\eta = \alpha + \beta$. The function $\tilde{w}$ satisfies a beautiful chain, being the discrete differential analogue of the Calogero–Degasperis equation (see (3)):

$$u_t = R(u)\left(\frac{1}{u_1 + u} - \frac{1}{u + u_{-1}}\right) \tag{18}$$

$$R(u) = (u^2 - 1)(\eta^2 - \mu^2 u^2). \tag{19}$$

Discrete Miura transformation of the chain (18), (19) into the modified Volterra equation (14) is given by

$$\tilde{u} = \frac{\mu u_1 u + \frac{1}{2}(\eta - \mu)(u_1 - u) + \eta}{u_1 + u}. \tag{20}$$

The chain (18), (19) has been obtained in [9] (see also the introduction of [10]). In [9] a complete list was given of scalar chains of the form $u_t = f(u_1, u, u_{-1})$ possessing an infinite set of local conservation laws. The author constructed conservation laws using transformations similar to (20), which were found by complicated calculations. Unfortunately, these transformations (and also (20)) are found only in the PhD thesis of the author.

3.2. Local conservation laws

Let us discuss the didactic example (18), (19) at greater length in order to demonstrate that the scheme permits one, starting with an integrable chain, to construct chains which are also integrable. In particular, if we start with a chain possessing local conservation laws, the new chain will have them as well. The local conservation law of the chain (4) is of the form $(\rho[u])_t = (D-1)(\sigma[u])$, where ρ, σ are scalar functions of a finite number of the variables u_i. It should be remarked that, in the case of the periodic closure $u_{i+N} = u_i$, we have a constant of the motion $h = \Sigma_1^N D^i(\rho)$, since $h_t = 0$. If a chain is reduced to (4) by a transformation $\tilde{u} = \varphi[u]$, this chain possesses the conservation law $(\rho[\varphi[u]])_t = (D-1)(\sigma[\varphi[u]])$.

Thus the Volterra equation (5) possesses conservation laws with densities

$$\rho^{(1)} = \tfrac{1}{2}\ln(u) \qquad \rho^{(2)} = u \qquad \rho^{(3)} = u_1 u + \tfrac{1}{2}u^2 \tag{21}$$

($\sigma^{(1)} = \frac{1}{2}(u + u_{-1})$, $\sigma^{(2)} = uu_{-1}$). The conserved density of (14) corresponding to $\rho^{(2)}$ is $\rho = (u+\alpha)(u_1 - \alpha)$. For the chain (18), (19) we have

$$\rho = \frac{(\mu u_2 - \eta)(u_1^2 - 1)(\mu u + \eta)}{(u_2 + u_1)(u_1 + u)}.$$

There exist simpler conserved densities of the chain (18), (19):

$$\ln\left(\frac{R(u)}{(u_1 + u)^2}\right) \qquad \int \frac{\varepsilon u^2 + \delta}{R(u)}\,du$$

where ε, δ are arbitrary constants.

Let us recall that the Volterra equation has an infinite set of local conservation laws because it admits Lax representation $L_t = AL - LA$ with

$$\begin{aligned} L &= u_1^{1/2} D + u^{1/2} D^{-1} \\ 2A &= u_2^{1/2} u_1^{1/2} D^2 - u^{1/2} u_{-1}^{1/2} D^{-2}. \end{aligned} \tag{22}$$

Operators of this kind are multiplied as follows: $(fD^k)(gD^m) = fD^k(g)D^{k+m}$. The function $\mathrm{res}(L^k)$ (namely the coefficient of L^k at D^0) is conserved density. For example, $\mathrm{res}(L^2) = u_1 + u \sim 2\rho^{(2)}$, $\mathrm{res}(L^4) = u_2 u_1 + (u_1 + u)^2 + uu_{-1} \sim 4\rho^{(3)}$ (densities ρ, $\hat{\rho}$ are equivalent if $\rho - \hat{\rho} \in \mathrm{Im}(D-1)$). Thus, the chain (18), (19) also possesses an infinite set of local conservation laws.

3.3. *Higher symmetries*

There exists the possibility to construct higher symmetries of the new chain. If there exist symmetries of the chain (F), (G) (see scheme (1)) which are reduced to a symmetry of (T) by the same transformations, then the described scheme enables one to construct a chain which is a symmetry of the new chain (H).

As is known, the chains (5) and (14) are Hamiltonian. This means that we can easily obtain their higher symmetries. In the case of the modified Volterra equation (14), symmetries are given by

$$u_\tau = K\delta\rho/\delta u \tag{23}$$

$$K = (u^2 - \alpha^2)(D - D^{-1})(u^2 - \alpha^2) \tag{24}$$

where ρ is a conserved density. The formal variational derivative $\delta\rho/\delta u$ is the function $\Sigma_i\, \partial(D^i\rho)/\partial u$. For instance, if $\rho = (u+\alpha)(u_1 - \alpha)$, then $\delta\rho/\delta u = u_1 + u_{-1}$, and the formulae (23), (24) give the simplest higher symmetry. In the case of the Volterra equation

(5), there exist two Hamiltonian operators:

$$K^{(1)}=u(D-D^{-1})u \tag{25}$$

$$K^{(2)}=u[u_1D^2+(u_1+u)D-(u+u_{-1})D^{-1}-u_{-1}D^{-2}]u. \tag{26}$$

Let ρ be some conserved density of (5), and q be a conserved density of (14), constructed with the help of the discrete Miura transformation (15). It is not hard to verify that the symmetry (23), (24) with $\rho=q$ is reduced by (15) to the following chain:

$$u_\tau=(K^{(2)}+4\alpha^2K^{(1)})\delta p/\delta u. \tag{27}$$

This is a symmetry of equation (5). The construction scheme can be applied if (27) does not depend on the parameter α.

There are conserved densities of (5) $\rho^{(1)}, \rho^{(2)}, \ldots$, such that

$$K^{(2)}\delta\rho^{(i)}/\delta u=K^{(1)}\delta\rho^{(i+1)}/\delta u \qquad K^{(1)}\delta\rho^{(1)}/\delta u=0 \tag{28}$$

($\rho^{(1)}, \rho^{(2)}, \rho^{(3)}$ are given by (21); it will be explained below how to obtain the others). If

$$p=\rho^{(k)}-4\alpha^2\rho^{(k-1)}+(4\alpha^2)^2\rho^{(k-2)}-\ldots+(-4\alpha^2)^{k-1}\rho^{(1)}$$

then the symmetry (27) takes the form (23), (25) with $\rho=\rho^{(k+1)}$ (i.e. $u_\tau=K^{(1)}\delta\rho^{(k+1)}/\delta u$) and does not depend on α. In particular, the symmetry (23), (24) with

$$\rho=(u+\alpha)(u_1-\alpha)-2\alpha^2\ln[(u+\alpha)(u_1-\alpha)]$$

of the chain (14) is reduced by (15) to the symmetry (23), (25) with $\rho=\rho^{(3)}$ of (5), and we can use the construction scheme to obtain the simplest higher symmetry of (18), (19). Let us write it down for the chain (18) with

$$R(u)=au^4+bu^2+c \tag{29}$$

(a, b, c are arbitrary constants). This symmetry of (18), (29) has the form

$$u_\tau=R(u)(1-D^{-1})\frac{1}{u_1+u}(D+1)\left[\frac{R(u)}{(u_1+u)(u+u_{-1})}-au^2\right].$$

In this way we can construct an infinite hierarchy of higher symmetries of the chain (18), (19).

Returning to the relationships (28), we denote $u\delta\rho^{(i)}/\delta u$ by $h^{(i)}$. The simplest way to find $h^{(i)}$ is to introduce the formal infinite series $Q=h^{(1)}+h^{(2)}\lambda^{-1}+h^{(3)}\lambda^{-2}+\ldots$ [11]. Equations (28) are equivalent to

$$u^{-1}(K^{(2)}-\lambda K^{(1)})(u^{-1}Q)=0. \tag{30}$$

Applying the operator

$$(D-1)^{-1}[D(Q)+Q](1+D^{-1})^{-1}$$

to (30), we can see that (30) is equivalent to

$$u[D(Q)+Q][Q+D^{-1}(Q)]=\lambda Q^2+c(\lambda) \tag{31}$$

where $c(\lambda)$ is a λ-dependent constant of integration. The coefficients $h^{(i)}$ can be explicitly found using (31). Setting $c(\lambda)=-\lambda/4$, $h^{(1)}=\frac{1}{2}$, we obtain $h^{(2)}=u$, $h^{(3)}=u(u_1+u+u_{-1})$. It can be proved that $h^{(i)}(i\geqslant 2)$ are homogeneous polynomials, such that $u\delta h^{(i)}/\delta u=(i-1)h^{(i)}$. As conserved densities $\rho^{(i)}$ satisfying (28) we may take the polynomials

$(i-1)^{-1}h^{(i)}$. Note that $h^{(2)}=\rho^{(2)}$ of (21), $\frac{1}{2}h^{(3)}\sim\rho^{(3)}$ (formal variational derivatives of equivalent conserved densities coincide with each other).

3.4. Soliton solutions

Let us discuss the problem of the construction of solutions. If one can construct solutions of the system (6), (8), then one can obtain solutions of the new chain (12) by means of formula (10). In the case of the chain (18), (19), this system (6), (8) becomes one consisting of (14), (16) and the following equation:

$$v_t=(v^2-\beta^2)(v_1-v_{-1}). \tag{32}$$

The constraint (16) is the Bäcklund transformation for the modified Volterra equation (14), and, therefore, it is possible to construct multi-soliton solutions of the system (14), (16), (32).

In order to make the formulae simpler, let us pass, by means of the point transformation $\tilde{u}=u+\alpha$, $\tilde{v}=v+\beta$, $\tilde{t}=-t$, from (14), (16), (32) to

$$u_t=(2\alpha-u)u(u_1-u_{-1}) \tag{33}$$

$$v_t=(2\beta-v)v(v_1-v_{-1}) \tag{34}$$

$$(2\alpha-u_1)u=(2\beta-v_1)v \tag{35}$$

and, by $\tilde{u}=u^{-1}$, $\tilde{t}=-t$, from (18), (19) to (18) with

$$R(u)=(u^2-1)[(\alpha+\beta)^2u^2-(\alpha-\beta)^2]. \tag{36}$$

It follows from the construction scheme that one can use not only the formula (10) but also the following one (see (9)): $w_{n+1}=q(u_n, v_n)$. Therefore there is the transformation

$$\tilde{u}=(u-v)/(u+v) \tag{37}$$

of (33)–(35) into (18), (36). This transformation will allow us to construct solutions with special properties. First we shall write down solutions for the system of equations (5) and

$$L_u\varphi=\lambda\varphi \qquad \varphi_t=A_u\varphi \tag{38}$$

(L_u, A_u are given by (22)). Then we shall use the fact that the function

$$\tilde{u}=u^{1/2}\varphi/\varphi_{-1} \qquad (\varphi_i=D^i(\varphi)) \tag{39}$$

satisfies (33) with $2\alpha=\lambda$, and the equality $(2\alpha-\tilde{u}_1)\tilde{u}=u$ takes place.

The dressing method gives the following real solutions $u^{(i)}$ ($i=1, 2, 3, \ldots$) of the Volterra equation (5) and $\varphi^{(i)}(\varepsilon_i, \nu_i)$, $\psi^{(i)}(\varepsilon_i, \nu_i)$ of (38) with $u=u^{(i)}$, $\lambda=2\cosh(\varepsilon_i)$ (here ε_i, ν_i are real parameters, $0<\varepsilon_1<\varepsilon_2<\varepsilon_3<\ldots$). If $i=1$, then $u^{(1)}=1$,

$$\begin{aligned} &\varphi^{(1)}(\varepsilon_1, \nu_1)=\exp[y_n(\varepsilon_1, \nu_1)] \\ &\psi^{(1)}(\varepsilon_1, \nu_1)=\exp[-y_n(\varepsilon_1, \nu_1)] \\ &y_n(\varepsilon, \nu)=n\varepsilon+t\sinh(2\varepsilon)+\nu \end{aligned} \tag{40}$$

where n is the discrete parameter. Also

$$u^{(i+1)}=(u_1^{(i)})^{1/2}(u_{-1}^{(i)})^{1/2}\theta_1^{(i)}\theta_{-2}^{(i)}/\theta^{(i)}\theta_{-1}^{(i)}$$

$$\varphi^{(i+1)}(\varepsilon_{i+1}, \nu_{i+1})=\Delta(u^{(i)}, \theta^{(i)})\varphi^{(i)}(\varepsilon_{i+1}, \nu_{i+1})$$

$$\psi^{(i+1)}(\varepsilon_{i+1}, \nu_{i+1})=-\Delta(u^{(i)}, \theta^{(i)})\psi^{(i)}(\varepsilon_{i+1}, \nu_{i+1})$$

where

$$\theta^{(i)}=\varphi^{(i)}(\varepsilon_i, \nu_i)+\psi^{(i)}(\varepsilon_i, \nu_i)$$

and Δ is the following operator:

$$\Delta(u, \varphi)=u_1^{1/4}u^{1/4}(\varphi_{-1}^{1/2}\varphi_1^{-1/2}D-\varphi_1^{1/2}\varphi_{-1}^{-1/2}D^{-1}).$$

Now we can easily construct solutions of the system (33)–(35). For any fixed i, we take functions of the form

$$\Phi=c_1\varphi^{(i)}(\varepsilon_i, \nu_i)+c_2\psi^{(i)}(\varepsilon_i, \nu_i)$$

$$\Psi=c_3\varphi^{(i)}(\delta_i, \eta_i)+c_4\psi^{(i)}(\delta_i, \eta_i)$$

where $c_k\geqslant 0$ are constants, $c_1+c_2\neq 0$, $c_3+c_4\neq 0$; δ_i, $\eta_i\in\mathbb{R}$, $\delta_i>\varepsilon_{i-1}(\delta_i>0$ if $i=1)$. The functions

$$\tilde{u}=(u^{(i)})^{1/2}\Phi/\Phi_{-1} \qquad \tilde{v}=(u^{(i)})^{1/2}\Psi/\Psi_{-1}$$

satisfy (33)–(35) with $\alpha=\cosh(\varepsilon_i)$, $\beta=\cosh(\delta_i)$ (see (39)). In accordance with (37), the following formula

$$(\Phi/\Phi_{-1}-\Psi/\Psi_{-1})/(\Phi/\Phi_{-1}+\Psi/\Psi_{-1})$$

yields solutions of the chain (18), (36). One can prove that $u^{(i)}$, $\varphi^{(i)}(\varepsilon_i, \nu_i)$, $\psi^{(i)}(\varepsilon_i, \nu_i)$ are positive for any $n\in\mathbb{Z}$, $t\in\mathbb{R}$ and do not have singularities. It is clear that the solutions of the chain (18), (36) also do not have singularities. They are also bounded.

Let $i=1$, $\Phi=\cosh y_n(\varepsilon_1, \nu_1)$, $\Psi=\exp y_n(\delta_1, \eta_1)$ (see (40)). We obtain for the chain (18), (36) with $\alpha=\cosh(\varepsilon_1)$, $\beta=\cosh(\delta_1)$ the following solution:

$$u_n=\frac{\cosh y_n(\varepsilon_1, \nu_1)-\exp(\delta_1)\cosh y_{n-1}(\varepsilon_1, \nu_1)}{\cosh y_n(\varepsilon_1, \nu_1)+\exp(\delta_1)\cosh y_{n-1}(\varepsilon_1, \nu_1)}.$$

Here $u_n\to\tanh[(\pm\varepsilon_1-\delta_1)/2]$ as $n\to\pm\infty$. We are led to another not very complicated example if $i=2$ and $\Phi=\varphi^{(2)}(\varepsilon_2, \nu_2)$, $\Psi=\varphi^{(2)}(\delta_2, \eta_2)$. In this case $\alpha=\cosh(\varepsilon_2)$, $\beta=\cosh(\delta_2)$, and $u_n=(p-q)/(p+q)$ with

$$p=\exp(\varepsilon_2)\frac{\cosh[y_n(\varepsilon_1, \nu_1)+z(\varepsilon_2, \varepsilon_1)]}{\cosh[y_{n-1}(\varepsilon_1, \nu_1)+z(\varepsilon_2, \varepsilon_1)]}$$

$$q=\exp(\delta_2)\frac{\cosh[y_n(\varepsilon_1, \nu_1)+z(\delta_2, \varepsilon_1)]}{\cosh[y_{n-1}(\varepsilon_1, \nu_1)+z(\delta_2, \varepsilon_1)]}$$

$$z(\varepsilon, \varepsilon_1)=\tfrac{1}{2}\ln[\sinh(\varepsilon-\varepsilon_1)/\sinh(\varepsilon+\varepsilon_1)].$$

Now $u_n\to\tanh[(\varepsilon_2-\delta_2)/2]$ as $n\to\pm\infty$.

3.5. *Zero-curvature representation*

After what has been said above, it is not very surprising that the chain (18), (29) has the zero-curvature representation $L_t = D(A)L - LA$, where

$$L = (R(u))^{-1/2}\begin{pmatrix} f(\lambda)u & c\lambda^{-1} - \lambda u^2 \\ a\lambda^{-1}u^2 - \lambda & f(\lambda)u \end{pmatrix}$$

$$A = \frac{1}{u+u_{-1}}\begin{pmatrix} g(\lambda)(u-u_{-1}) & f(\lambda)(\lambda u u_{-1} - c\lambda^{-1}) \\ f(\lambda)(\lambda - a\lambda^{-1}uu_{-1}) & -g(\lambda)(u-u_{-1}) \end{pmatrix}$$

$$(f(\lambda))^2 = \lambda^2 + ac\lambda^{-2} + b \qquad 2g(\lambda) = \lambda^2 - ac\lambda^{-2}.$$

Although the function $f(\lambda)$ is not rational, those who wish can easily obtain rational dependence on the spectral parameter λ (and polynomial dependence for the matrix L).

4. Other examples

In this section we consider systems of two discrete differential equations, related to the Toda model. It is demonstrated, in particular, that there exist many instances in which the proposed approach can be applied. Also, a useful and elementary addition to the construction scheme is discussed.

4.1. *Toda-model example*

There are quite a lot of cases in which we can use the construction scheme. To be convinced of this, let us consider a rich example related to the polynomial Toda chain

$$u_t = u(v_1 - v) \qquad v_t = u - u_{-1}. \tag{41}$$

There exist seven chains and eight transformations of the form

$$\tilde{u} = \tilde{u}(u, v, u_1, v_1) \qquad \tilde{v} = \tilde{v}(u, v, u_1, v_1) \tag{42}$$

reducing them to (41):

$$u_t = u(v_1 - v) \qquad v_t = v(u - u_{-1})$$

$$\tilde{u} = uv_1 \qquad \tilde{v} = u + v \tag{43}$$

$$\tilde{u} = u_1 v_1 \qquad \tilde{v} = u + v_1$$

$$\begin{aligned} &u_t = v \qquad v_t = \exp(u_1 - u) - \exp(u - u_{-1}) \\ &\tilde{u} = \exp(u_1 - u) \qquad \tilde{v} = v \end{aligned} \tag{44}$$

$$\begin{aligned} &u_t = u(v_1 - 2v + v_{-1}) \qquad v_t = u \\ &\tilde{u} = u_1 \qquad \tilde{v} = v_1 - v \end{aligned} \tag{45}$$

$$\begin{aligned} &u_t = \exp(v_1 - u) + u - v \qquad v_t = u - v + \exp(v - u_{-1}) \\ &\tilde{u} = \exp(v_1 - u) \qquad \tilde{v} = u - v \end{aligned} \tag{46}$$

$$\begin{aligned} &u_t = v_1 - u + \exp(u - v) \qquad v_t = \exp(u - v) + v - u_{-1} \\ &\tilde{u} = \exp(u_1 - v_1) \qquad \tilde{v} = v_1 - u \end{aligned} \tag{47}$$

$$u_t=(v_1-u)(u-v)^{1/2} \qquad v_t=(u-v)^{1/2}(v-u_{-1})$$
$$2\tilde{u}=v_1-u \qquad \tilde{v}=(u-v)^{1/2} \tag{48}$$

$$u_t=(v_1-u)^{1/2}(u-v) \qquad v_t=(u-v)(v-u_{-1})^{1/2}$$
$$2\tilde{u}=u_1-v_1 \qquad \tilde{v}=(v_1-u)^{1/2}. \tag{49}$$

The integrable discrete differential model found by Toda is equation (44). Equation (43) is equivalent to the Volterra equation (5) ((5) is turned into (43) by $\tilde{u}_n=u_{2n}$, $\tilde{v}_n=u_{2n-1}$).

Considering transformations corresponding to (43)–(49), one can check that there are 14 pairs of transformations which satisfy condition 2. That is to say we have 14 possibilities to use the scheme. For example, taking the Volterra equation (43) (together with the second of the corresponding transformations) and the Toda model (44), we obtain the following chain and transformations:

$$u_t=(v_1)^{-1}+u^2v \qquad -v_t=(u_{-1})^{-1}+v^2u$$
$$(50)\to(43):\tilde{u}=u^{-1}v_1^{-1} \qquad \tilde{v}=uv \tag{50}$$
$$(50)\to(44):\tilde{u}=-\ln(v_1) \qquad \tilde{v}=u^{-1}v_1^{-1}+u_1v_1.$$

Equations (43)–(49) can be called chains of the first level. The scheme enables us to obtain chains of the fourth level (one needs four transformations of the form (42) to reduce such a chain to (41)). Each of the levels contains more than enough chains. So there exist five chains which are reduced to the Volterra equation (43) by transformations of the form (42). In this case there are eight transformations again and a lot of possibilities to apply the scheme. One of these five chains generalizes the modified Volterra equation (14):

$$u_t=(u^2-\alpha^2)(v_1-v) \qquad v_t=(v^2-\beta^2)(u-u_{-1}). \tag{51}$$

Transformations reducing (51) to (43) are given by

$$\tilde{u}=(u+\alpha)(v_1+\beta) \qquad \tilde{v}=(u-\alpha)(v-\beta)$$
$$\tilde{u}=(u_1+\alpha)(v_1+\beta) \qquad \tilde{v}=(u-\alpha)(v_1-\beta).$$

Using (51) and these transformations, we can construct a generalization of the discrete differential analogue of the Calogero–Degasperis equation (18), (19).

4.2. *Zeroth-order conservation laws*

We can see that it is an interesting problem to describe all chains which are reduced to the polynomial Toda chain (41). The construction scheme can be very helpful in this connection, however, it does not enable one to obtain all chains of this kind. In particular, we are not able to obtain such chains as those of the first level ((43)–(49)) because there are no transformations in this case.

We can construct new discrete differential systems and corresponding transformations of the form (42), using not only the proposed scheme but also local conservation laws of the zeroth order: $\rho_t=(D-1)(\sigma)$, where $\rho=\rho(u,v)$ (as regards partial differential equations, see [10, 12]). For example, conserved densities of the form $\rho(u, v)$ for the polynomial Toda chain (41) are described by the following formula:

$$\rho=\alpha(v^2+2u)+\beta v+\gamma \ln(u)+\delta \tag{52}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary constants (if $\rho=v^2+2u$, then $\sigma=2vu_{-1}$). We can introduce new variables U_i, V_i as follows: $U_1-U=\rho(u, v)$, $V=r(u, v)$, where ρ and r are functionally independent. Formulae of the form

$$u=u(U_1-U, V) \qquad v=v(U_1-U, V) \tag{53}$$

occur. Setting $U_t=\sigma$, $V_t=r_t$ and using (53), we easily obtain a chain in terms of U_i, V_i, which is reduced to the given chain by the transformation (53). Then we can simplify the chain resulting by point transformations: $U=U(\tilde{U}, \tilde{V})$, $V=V(\tilde{U}, \tilde{V})$. It is an easy matter to construct in this way (44), (45), and (48), using conserved densities (52).

4.3. *Schrödinger-type systems*

It is worthwhile to remark that, if we construct integrable chains similar to (41) together with discrete transformations, we can obtain, at the same time, integrable partial differential systems of the Schrödinger type (which were investigated and classified in [10, 12]) together with transformations relating them [13, 14]. Let us consider an example of how Miura-type transformations of chains generate chains of corresponding Schrödinger-type systems.

There correspond the well known systems

$$U_\tau=U_{tt}+(2UV)_t \qquad V_\tau=-V_{tt}+(2U+V^2)_t \tag{54}$$

$$U_\tau=U_{tt}+(2UV+U^2)_t \qquad V_\tau=-V_{tt}+(2UV+V^2)_t \tag{55}$$

to the polynomial Toda chain (41) and Volterra equation (43), respectively [15]. To obtain Miura transformations in this case, the solutions u_n, v_n of (41) (or (43)) must satisfy the system (54) (or (55)) for any integer n. The first of the transformations of (43) into (41) can be rewritten in the following way: $\tilde{u}_n=u_nv_{n+1}=u_{nt}+u_nv_n$, $\tilde{v}_n=u_n+v_n$. In the second case we have: $\tilde{u}_n=u_{n+1}v_{n+1}$, $\tilde{v}_n=u_n+v_{n+1}=-(\ln v_{n+1})_t+u_{n+1}+v_{n+1}$. We see that the system (55) is reduced to the dispersive water waves equation (54) by

$$\tilde{U}=U_t+UV \qquad \tilde{V}=U+V \tag{56}$$

$$\tilde{U}=UV \qquad \tilde{V}=U+V-(\ln V)_t. \tag{57}$$

It is possible that the transformations (56) and (57) have not previously been known.

Using results from [14] and discrete transformations, one can construct many other transformations of Schrödinger-type systems. There are many transformations of this kind in [12], however, the purpose of that paper was not to find all the transformations.

5. Conclusions

5.1. *Open problem*

Recall that the analogue of the Calogero–Degasperis equation (18), (19) is reduced to the modified Volterra equation (14) by the discrete Miura transformation (20). We may try to obtain one more integrable chain. As in section 3.1, we obtain the following constraint:

$$\frac{(\alpha-\beta)u_1u+\beta(u_1-u)+\alpha+\beta}{u_1+u}=\frac{(\alpha-\gamma)v_1v+\gamma(v_1-v)+\alpha+\gamma}{v_1+v}. \tag{58}$$

The problem is now to express this constraint in the form (9) (see condition 2). On the fact of it, this is not so easy as in the case of (16), nevertheless, this can be done. If $\alpha\neq\beta$, then (58) is equivalent to

$$(v-1)\frac{\alpha+\beta+(\alpha-\beta)u}{(\alpha-\gamma)v-(\alpha-\beta)u-(\beta-\gamma)}=(v_1+1)\frac{\alpha+\beta-(\alpha-\beta)u_1}{(\alpha-\gamma)v_1-(\alpha-\beta)u_1+(\beta-\gamma)}.$$

So we can use the scheme and construct a complicated integrable chain (which first appeared in [9]) and a corresponding Miura-type transformation.

An interesting open problem is to describe relationships of the form (8), which can be expressed in the form (9) (remember that the vector case is considered).

5.2. *Other approaches*

As is shown in many papers of Ufa mathematicians, concerning the classsification of integrable equations (see [10] for details and references), most integrable equations are reduced to a few simple enough equations by transformations which can be called differential substitutions in the case of partial differential equations (56), (57), and discrete substitutions in the case of discrete differential equations (see all the other transformations in this paper). This indicates the necessity for a well developed theory of transformations of differential equations (in particular, a theory of differential and discrete substitutions). There are several different approaches to this problem at present (we discuss here only substitutions).

In the first place, as we know already, starting with some key integrable equation, one can construct other integrable equations and substitutions by the scheme presented in this paper and in [1]. One can also use local conservation laws of the zeroth order. In many cases, Bäcklund auto-transformations enable us to construct substitutions and to obtain new equations [2]. These means prove to be convenient if we start with an equation integrable by the inverse scattering method. One may consider that here we go from lower equations to upper ones (see scheme (1)).

There are other possibilities to go in the same direction and to obtain substitutions. A method presented in [16] uses *L–A* pairs and gives formulae of the form (39) first of all. The well known method of the factorization of differential operators also uses *L–A* pairs and allows one to obtain good results in many cases (see [11] for references; see also [17]).

If we start with some linear equation possessing a rich enough Lie algebra of classical symmetries (the heat equation, for instance), an approach developed in [18, 19] will be convenient. In order to construct integrable equations and substitutions, it uses classical symmetries. In this case, one goes in the opposite direction: from upper equations to lower ones. It is possible to move in the same direction by so-called pseudo-symmetries and special conservation laws [18]. It is interesting that Bäcklund auto-transformations enable us to construct substitutions, starting from both lower and upper equations [2].

There exists another approach which gives so-called symmetrical transformations (these are compositions of differential substitutions of a special form) [10, 12]. Symmetrical transformations are constructed for equations possessing both a classical symmetry and a local conservation law of the zeroth or first order.

It should be said that most of these papers are devoted to partial differential equations and differential substitution. However, the schemes and methods can easily be used in the discrete differential case.

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